BELYI MAPS AND DESSINS D'ENFANTS
LECTURE 15

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I. REVIEW

Last time we:

(1) Given a Fuchsian group Γ and a fundamental domain D for Γ, showed how we can obtain a fundamental domain for any subgroup as a union of translates of D.
(2) Applied this in the particular case of $\Gamma = \Gamma(1) = \mathrm{PSL}_2(\mathbb{Z})$ and the subgroup $\Gamma(2)$.
(3) Specialized some results on covering spaces and monodromy to the particular case of the covering $\mathfrak{H} \to \Gamma\backslash\mathfrak{H}$ where Γ is a Fuchsian group.
(4) In particular, used Riemann-Hurwitz to give a formula for the genus of $\Gamma(N)\backslash\mathfrak{H}$.

II. MONODROMY AND FUCHSIAN GROUPS

II.1. **Monodromy via Fuchsian groups.** Recall that, given a morphism $F : X \to Y$, the monodromy of F describes the action of the fundamental group $\pi_1(Y)$ on a fiber of F. We can reinterpret this in terms of Fuchsian groups as well.

Let $F : X \to Y$ be a morphism of Riemann surfaces and let $B \subseteq Y$ be its set of ramification values. Let $Y^* = Y \setminus B$, and let $X^* = F^{-1}(Y^*)$ so $F|_{X^*} : X^* \to Y^*$ is an unramified covering. Applying the uniformization theorem, we obtain Fuchsian groups $\Gamma_X \leq \Gamma_Y$ such that

$$X^* \cong \Gamma_X\backslash\mathfrak{H} \qquad Y^* \cong \Gamma_Y\backslash\mathfrak{H}$$

as well as a morphism $G : \Gamma_X\backslash\mathfrak{H} \to \Gamma_Y\backslash\mathfrak{H}$ such that the following diagram commutes

Date: May 17, 2021.

$$\begin{CD} X^* @>\sim>> \Gamma_X\backslash\mathfrak{H} \\ @V{F|_{X^*}}VV @VV{G}V \\ Y^* @>\sim>> \Gamma_Y\backslash\mathfrak{H} \end{CD}$$

Since $\mathfrak{H} \to \Gamma_Y\backslash\mathfrak{H} \cong Y^*$ is the universal cover of Y^* and $\mathrm{Deck}(\mathfrak{H} \to Y^*) \cong \Gamma_Y$, then

$$\pi_1(Y^*) \cong \Gamma_Y .$$

Given $y \in Y$, then y corresponds to $[z_0]_{\Gamma_Y} \in \Gamma_Y\backslash\mathfrak{H}$ for some $z_0 \in \mathfrak{H}$. (Here $[\cdot]_{\Gamma_Y}$ denotes the equivalence class modulo the action of Γ_Y.) Moreover, by commutativity of the diagram

$$\begin{array}{ccc} \mathfrak{H} & & \\ {\scriptstyle\varphi_X}\big\downarrow & \searrow^{\varphi_Y} & \\ \Gamma_X\backslash\mathfrak{H} & \xrightarrow[G]{} & \Gamma_Y\backslash\mathfrak{H} \end{array} \tag{1}$$

given $y \in Y$, the fiber $G^{-1}(y)$ is

$$\{[\beta(z_0)]_{\Gamma_X} : \beta \in \Gamma_X\backslash\Gamma_Y\}$$

where β ranges over a set of right coset representatives for $\Gamma_X\backslash\Gamma_Y$. Thus we have a bijection

$$\begin{aligned} \Phi : \Gamma_X\backslash\Gamma_Y &\to G^{-1}(y) \\ \Gamma_X\beta &\mapsto [\beta(z_0)]_{\Gamma_X} . \end{aligned} \tag{2}$$

We want to reinterpret the monodromy representation in terms of the groups Γ_X and Γ_Y using the above bijection. Let

$$\rho : \pi_1(Y) \to \mathrm{Sym}(G^{-1}(y))$$

be the monodromy representation of G. Given $\gamma \in \Gamma_Y$, we describe the action of the permutation $\rho(\gamma)$ on $\Gamma_X\backslash\Gamma_Y$ as follows.

Under the isomorphism $\Gamma \cong \pi_1(Y^*)$, γ corresponds to a loop c on Y^*. Choose a lift $\widetilde{c}_{\mathfrak{H}}$ of c to the universal cover $\mathfrak{H}$ starting at the point z_0, and its terminal point is $\gamma(z_0)$. On the other hand, we can also lift c to the cover $X^* \cong \Gamma_X\backslash\mathfrak{H}$: let $\widetilde{c}_X$ be the lift of c to X^* with initial point $[\beta(z_0)]_{\Gamma_X}$. We now determine the relationship between $\widetilde{c}_{\mathfrak{H}}$ and $\widetilde{c}_X$.

$\widetilde{c}_{\mathfrak{H}}$

$$\begin{array}{ccc} \mathfrak{H} & & \\ {\scriptstyle\varphi_X}\big\downarrow & \searrow^{\varphi_Y} & \\ \Gamma_X\backslash\mathfrak{H} & \xrightarrow[G]{} & \Gamma_Y\backslash\mathfrak{H} \end{array} \tag{3}$$

$\widetilde{c}_X$ $\qquad$ c

Consider the path $\beta \circ \widetilde{c}_{\mathfrak{H}}$ on $\mathfrak{H}$, which has initial point $\beta(z_0)$ and terminal point $\beta\gamma(z_0)$. (Note that $\beta \in \Gamma_X \subseteq \mathrm{Aut}(\mathfrak{H}) \cong \mathrm{PSL}_2(\mathbb{R})$, so β is an automorphism of $\mathfrak{H}$.) Then $\varphi_X \circ \beta \circ \widetilde{c}_{\mathfrak{H}}$ is a path on $X \cong \Gamma_X\backslash\mathfrak{H}$ with initial point $[\beta(z_0)]_{\Gamma_X}$ and terminal point $[\beta\gamma(z_0)]_{\Gamma_X}$.

Since $\varphi_X \circ \beta \circ \widetilde{c}_{\mathfrak{H}}$ and $\widetilde{c}_X$ are both lifts of c to X starting at the point $[\beta(z_0)]_{\Gamma_X}$, by commutativity of the diagram (3), we have $\varphi_X \circ \beta \circ \widetilde{c}_{\mathfrak{H}} = \widetilde{c}_X$. In particular, then they have the same terminal points, so the terminal point of $\widetilde{c}_X$ is $[\beta\gamma(z_0)]_X$.

Under the correspondence between the fiber $G^{-1}(y)$ and the coset space $\Gamma_X \backslash \Gamma_Y$ given by (2), the coset $\Gamma_X\beta$ is mapped to $\Gamma_X\beta\gamma$. However(!), recall that this gives a right action on the fiber. In order to obtain the associated left action, we must take an inverse. Thus $\rho(\gamma)$ maps $\Gamma_X\beta \mapsto \Gamma_X\beta\gamma^{-1}$ and the diagram below commutes.

$$\begin{array}{ccc} x & \longmapsto & \rho(\gamma)\, x \\ G^{-1}(y) & \xrightarrow{\rho(\gamma)} & G^{-1}(y) \\ \Phi \big\uparrow & & \Phi \big\uparrow \\ \Gamma_X \backslash \Gamma_Y & \xrightarrow[\rho(\gamma)]{} & \Gamma_X \backslash \Gamma_Y \\ \Gamma_X\beta & \longmapsto & \Gamma_X\beta\gamma^{-1} \end{array}$$

Lemma 1. *With notation as above, the stabilizer of a coset $\Gamma_X\beta \in \Gamma_X \backslash \Gamma_Y$ is*

$$\operatorname{Stab}_{\Gamma_Y}(\Gamma_X\beta) = \{\gamma \in \Gamma_Y : \Gamma_X\beta = \Gamma\beta\gamma\} = \beta^{-1}\Gamma_X\beta\,.$$

Proof. Given an element $\beta^{-1}\gamma\beta \in \beta^{-1}\Gamma_X\beta$, then

$$\Gamma_X\beta\,\beta^{-1}\gamma\beta = \Gamma_X\gamma\beta = \Gamma_X\beta$$

so $\beta^{-1}\gamma\beta$ stabilizes $\Gamma_X\beta$.

Conversely, suppose $\gamma \in \Gamma_Y$ stabilizes $\Gamma_X\beta$, so $\Gamma_X\beta\gamma^{-1} = \Gamma_X\beta$. Then $\Gamma_X = \Gamma_X\beta\gamma\beta^{-1}$, so $\beta\gamma\beta^{-1} \in \Gamma_X$. Thus

$$\gamma = \beta^{-1}(\beta\gamma\beta^{-1})\beta \in \beta^{-1}\Gamma_X\beta\,.$$

□

Remark 2. Taking $\beta = 1$, then in particular

$$\Gamma_X = \operatorname{Stab}_{\Gamma_Y}(\Gamma_X)\,.$$

This shows how to recover Γ_X from the monodromy representation as the stabilizer of an element of the fiber. For instance, suppose $\varphi : X \to \mathbb{P}^1$ is a Belyi map. By the uniformization theorem, then there exists a triangle group $\Delta := \Delta(a,b,c)$ and a subgroup $\Gamma \leq \Delta$ such that $\mathbb{P}^1 \setminus \{0,1,\infty\} \cong \Delta\backslash\mathfrak{H}$ and $X^* \cong \Gamma\backslash\mathfrak{H}$, where $X^* = X \setminus \varphi^{-1}(\{0,1,\infty\})$. Choose a base point $z_0 \in \mathbb{P}^1 \setminus \{0,1,\infty\}$, and label the points of the fiber $\varphi^{-1}(z_0)$, so

$$\varphi^{-1}(z_0) = \{x_1, x_2, \ldots, x_d\}$$

where d is the degree of φ. By the above, this is equivalent to choosing a set of representatives $\Gamma\beta_1, \ldots, \Gamma\beta_d$ for the coset space $\Gamma\backslash\Delta$. Recall that $\pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\})$ has presentation

$$\langle \eta_0, \eta_1, \eta_\infty \mid \eta_\infty\eta_1\eta_0 = 1\rangle$$

where $\eta_0, \eta_1, \eta_\infty$ are homotopy classes represented by small loops around $0, 1$, and ∞, respectively. Let $\rho : \pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}) \to S_d$ be the monodromy representation of φ, and let $\sigma_0, \sigma_1, \sigma_\infty$ be the images of $\eta_0, \eta_1, \eta_\infty$ under ρ. Since X^* uniformized by $\Delta = \Delta(a,b,c)$, then $\sigma_0, \sigma_1, \sigma_\infty$ have orders a, b, c in S_d. Thus ρ descends to a homomorphism $\overline{\rho} : \Delta \to S_d$ such that the diagram below commutes.

$$\begin{array}{ccc} \pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}) & \xrightarrow{\rho} & S_d \\ \downarrow & \overset{\overline{\rho}}{\nearrow} & \\ \Delta & & \end{array}$$

By the lemma above, given just the permutations $\sigma_0, \sigma_1, \sigma_\infty$, we can recover Γ as

$$\Gamma = \mathrm{Stab}_\Delta(1)$$

where Δ acts on $\{1, \ldots, d\}$ via the identification with the set $\{\Gamma\beta_1, \ldots, \Gamma\beta_d\}$, or equivalently, via the homomorphism

$$\begin{aligned} \overline{\rho} : \Delta &\to S_d \\ \delta_a, \delta_b, \delta_c &\mapsto \sigma_0, \sigma_1, \sigma_\infty \,. \end{aligned}$$

Remark 3. If we chose a different numbering for the coset representatives, we would obtain a conjugate (and hence isomorphic) subgroup Γ as the stabilizer of 1.

III. Galois coverings and morphisms, revisited

III.1. **Review.** Recall the definition of a Galois morphism of Riemann surfaces.

Definition 4. Let $F : X \to Y$ be a morphism of Riemann surfaces with ramification values $B \subseteq Y$. Then F is Galois if for each $y \in Y$, $\mathrm{Deck}(X \xrightarrow{F} Y)$ acts transitively on the fiber $F^{-1}(y)$.

We also gave a field theoretic characterization of Galois morphisms. Let L/K be a finite extension of fields. Recall that the following are equivalent.

(1) L/K is Galois.
(2) L/K is normal and separable.
(3) $\# \mathrm{Aut}(L/K) = [L : K]$.
(4) $L^{\mathrm{Aut}(L/K)} = K$, i.e., the fixed field of $\mathrm{Aut}(L/K)$ is K.

Given a morphism $F : X \to Y$ of Riemann surfaces, there is an induced extension of function fields:

$$\begin{aligned} F^* : \mathcal{M}(Y) &\hookrightarrow \mathcal{M}(X) \\ h &\mapsto h \circ F \,. \end{aligned}$$

Proposition 5. *A morphism $F : X \to Y$ of Riemann surfaces is Galois iff the induced function field extension $\mathcal{M}(X)/F^*\mathcal{M}(Y)$ is Galois.*

III.2. **More topology.** Let $(X, x), (Y, y)$ be pointed topological spaces, and let $F : (X, x) \to (Y, y)$ be a continuous map of pointed spaces (i.e., a continuous map $F : X \to Y$ with $F(x) = y$). Given a loop $\gamma : [0, 1] \to X$, then

$$[0,1] \xrightarrow{\gamma} X \xrightarrow{F} Y$$

is a loop in Y. One can show that this descends to a map

$$F_* : \pi_1(X, x) \to \pi_1(Y, y)$$
$$[\gamma] \mapsto [F \circ \gamma]$$

on fundamental groups. Moreover, F_* is a homomorphism of groups.

Proposition 6. *Let $F : (\widetilde{X}, \widetilde{x}) \to (X, x)$ be a covering map. Then the induced map*

$$F_* : \pi_1(\widetilde{X}, \widetilde{x}) \to \pi_1(X, x)$$

is injective.

Proof. This basically follows from the uniqueness of liftings of paths and homotopies to a covering space. Suppose $\widetilde{\gamma}$ is a loop in $\widetilde{X}$ such that

$$[c] = 1 = F_*([\widetilde{\gamma}]) = [F \circ \widetilde{\gamma}]\,,$$

where c is the constant path at x. Then there exists a homotopy H from $F \circ \widetilde{\gamma}$ to c. By homomotopy lifting, this lifts to a homotopy $\widetilde{H}$ from $\widetilde{\gamma}$ to $\widetilde{c}$, where $\widetilde{c}$ is the constant path at $\widetilde{x}$. Thus

$$[\widetilde{\gamma}] = [\widetilde{c}] = 1\,.$$

□

Recall that $\pi_1(X, x)$ acts on the fiber $F^{-1}(x)$ by path lifting, and this action is how the monodromy group is defined.

Proposition 7. *Let $F : \widetilde{X} \to X$ be a covering map of topological spaces and assume that $\widetilde{X}$ is path-connected. The for each $\widetilde{x} \in F^{-1}(x)$,*

$$\mathrm{Stab}_{\pi_1(X,x)}(\widetilde{x}) = F_*\pi_1(\widetilde{X}, \widetilde{x})\,.$$

Proof. Given $[\gamma] \in \pi_1(X, x)$, let $\widetilde{\gamma}$ be the path lift of γ to $\widetilde{X}$ starting at $\widetilde{x}$. Since $[\gamma]$ stabilizes $\widetilde{x}$, then $\widetilde{\gamma}$ has endpoint $\widetilde{x}$, as well. Thus $\widetilde{\gamma}$ is a loop in $\widetilde{X}$, so $[\widetilde{\gamma}] \in \pi_1(\widetilde{X}, \widetilde{x})$, hence

$$[\gamma] = [F \circ \widetilde{\gamma}] = F_*([\widetilde{\gamma}]) \in F_*\pi_1(\widetilde{X}, \widetilde{x})\,.$$

Conversely, given $[\gamma] = F_*([\widetilde{\gamma}]) \in F_*\pi_1(\widetilde{X}, \widetilde{x})$, then $\widetilde{\gamma}$ is the (unique) lift of γ to $\widetilde{X}$ starting at $\widetilde{x}$. And since $\widetilde{\gamma}$ starts and ends at $\widetilde{x}$ (it's a loop), then $[\gamma]$ stabilizes $\widetilde{x}$. □

III.3. **Galois groups and monodromy groups.**

Proposition 8. *If $F : X \to Y$ is a Galois morphism of Riemann surfaces, then* $\mathrm{Deck}(X/Y) \cong \mathrm{Mon}(F)$.

Remark 9. We have previously seen that in this case, we have $\mathrm{Gal}(\mathcal{M}(X)/\mathcal{M}(Y)) \cong \mathrm{Deck}(X/Y)$, so

$$\mathrm{Gal}(\mathcal{M}(X)/\mathcal{M}(Y)) \cong \mathrm{Deck}(X/Y) \cong \mathrm{Mon}(F)\,.$$

Proposition 10. *Let $F : X \to Y$ be a morphism of Riemann surfaces. Then F is Galois iff $\deg(F) = \#\mathrm{Mon}(F)$.*

Proof. The reverse implication requires some results on lifts of maps to covering spaces.

□

¡++¿

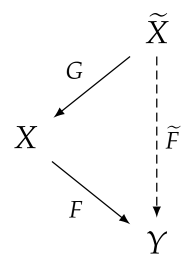
Even if $F : X \to Y$ is not Galois, we can "extend" it into a Galois morphism. We first recall the definition of Galois closure in the case of fields.

Definition 11. Let L/K be a separable extension of fields. The Galois closure of L/K is the smallest extension E of L, by inclusion, such that E is Galois.

Remark 12.

- Given two extensions E_1, E_2 of L such that E_1/K and E_2/K are both Galois, then $E_1 \cap E_2$ is also Galois. (Here this intersection is taken inside a fixed algebraic closure of K.) Thus there is a smallest such Galois extension, so the definition above makes sense.
- In the case where $L = K(\alpha)$ is a simple extension, then the Galois closure of L/K is simply the splitting field of the minimal polynomial of α.

Definition 13. The Galois closure or normal closure of a morphism $F : X \to Y$ is a Galois morphism $\widetilde{F} : \widetilde{X} \to Y$ of smallest possible degree, together with a morphism $G : \widetilde{X} \to X$ such that the following diagram commutes.



Remark 14. This is exactly the morphism corresponding to the Galois closure of the function field extension $\mathcal{M}(X)/F^*\mathcal{M}(Y)$.

Theorem 15. *Let $F : X \to Y$ be a morphism of Riemann surfaces, and let $\widetilde{F} : \widetilde{X} \to Y$ be its Galois closure. Then*

$$\mathrm{Mon}(F) \cong \mathrm{Deck}\left(\widetilde{X} \xrightarrow{\widetilde{F}} Y\right) \cong \mathrm{Gal}(\mathcal{M}(\widetilde{X})/\widetilde{F}^*\mathcal{M}(Y)).$$