BELYI PROJECT IDEAS

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(1) **Non-hyperelliptic Belyi maps of genus** 3. We have implemented our method so that Belyi maps of genera 0 and 1, as well as hyperelliptic Belyi maps, can be computed with the press of a button. We would like to implement a button for non-hyperelliptic genus 3 curves, too.

(2) **Newton's method for hyperelliptic curves**. Newton's method has been very useful for computing equations for Belyi maps on curves of genus 0 and 1. By writing down a system of polynomial equations the coefficients of the map must satisfy, and then applying Newton's method, we can quickly improve an approximate answer from 40 digits of precision to thousands. For maps defined over number fields of moderate to large degree, this extra precision is crucial in being abled to recognize the coefficients as algebraic numbers.

To implement this for hyperelliptic curves, we would take as input a permutation triple and write code to automatically output a system of polynomial equations the coefficients of the corresponding Belyi map satisfy.

(3) **Primitive Belyi maps**. The degree 12 Belyi map corresponding to the permutation triple

$$\begin{aligned}
\sigma_0 &= (1,8)(4,11)(5,10)(6,12)(7,9),\\
\sigma_1 &= (1,6,3,4)(2,5,8,7)(9,10,12,11),\\
\sigma_\infty &= (1,5,9,8,4,12)(2,7,11,3,6,10)
\end{aligned}$$

is

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> X;
Elliptic Curve defined by y^2 = x^3 - 12636/961*x + 408240/29791 over
    Rational Field
> phi;
(972/31*x^5 - 323676/961*x^4 + 46609344/29791*x^3 -
    3735675936/923521*x^2 + 164789225280/28629151*x -
    3450165990336/887503681)/(x^6 + 108/31*x^5 - 972/31*x^4 -
    2402784/29791*x^3 + 11022480/29791*x^2 + 13888324800/28629151*x -
    1458274104000/887503681)
```

Notice that there is no y in the equation for ϕ! So ϕ is really the pullback of a genus 0 Belyi map $\psi : \mathbb{P}^1 \to \mathbb{P}^1$ under the projection $(x,y) \mapsto x$.

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Using Magma and Sage, I was able to compute the monodromy group and permutation triple of ψ, and find the corresponding map in our database. The monodromy group of the original map ϕ is an imprimitive permutation group, and its "primitivization" is the monodromy group of ψ.

(a) Is there an easy way to see what the permutation triple corresponding to ψ should be without involved calculations?

(b) If we know equations for ψ, is there an easy way to "lift" these to equations for ϕ? (Mike Musty worked on this problem for 2-lifts in his thesis.)

(4) **Belyi maps arising as fiber products**. While trying to compute the Belyi map corresponding to the permutation triple

$$\sigma_0 = (1,2,3,4,5,6,7),$$
$$\sigma_1 = (1,3,6,4,5,2,7),$$
$$\sigma_\infty = (1,2,7,3,5,6,4)$$

we were able to compute the curve, $Y^2 = 16X^6 + 56X^4 + 17X^2 + 8$, but not the map. Noam remarked

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     Because the two bielliptic quotients are 3-isogenous,
the curve should have extra automorphisms, and indeed
the involution  X --> (2X+3) / (4X-2)  of the X-line lifts to
the genus-2 curve.  (This involution of P^1, together with
X --> -X, generates a group S_3.  As I told JV earlier this PM,
I suspect that we're dealing with a fiber product of the
quintic Belyi map P^1 --> P^1 with cycles 5, 32, 2111
with the S_3 Belyi map P^1 --> P^1.
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(a) Given a Belyi map $\varphi : X \to \mathbb{P}^1$ and a genus 0 Belyi map $\psi : \mathbb{P}^1 \to \mathbb{P}^1$ (i.e., given equations for both), can we determine equations for their fiber product?

(b) Conversely, given a Belyi map, can we determine if it arises as a fiber product?

(5) **Decomposing passports into refined passports**. A passport if the data of a transitive subgroup $G \leq S_d$ along with 3 conjugacy classes C_0, C_1, C_∞ in S_d. The collection of Belyi maps belonging to a passport is Galois stable, so a passport is a union of Galois orbits.

However, we can often find smaller Galois stable subsets by instead looking at conjugacy classes in G, rather than S_d. These are called refined passports. We would like to implement a method for splitting passports into refined passports.

(6) **Belyi maps in the LMFDB**. There is always development to be done on the collection of the Belyi maps in the LMFDB. A TODO list is available here: `https://github.com/LMFDB/lmfdb/blob/master/lmfdb/belyi/belyi_TODO.txt`