18.700 PROBLEM SET 5

Due Wednesday, October 23 at 11:59 pm on Canvas

Collaborated with:
Sources used:

Problem 1. (3D #18) (8 points) Show that V and $\mathcal{L}(\mathbb{F}, V)$ are isomorphic $\mathbb{F}$-vector spaces.

Solution. Observe that $\dim \mathbb{F} = 1$ because the multiplicative identity is a basis for $\mathbb{F}$. Proposition 3.72 from the textbook tells us that $\dim \mathcal{L}(\mathbb{F}, V) = (\dim \mathbb{F})(\dim V) = \dim V$. Since $V, \mathcal{L}(\mathbb{F}, V)$ are both vector spaces, and we have just shown that they have the same (finite) dimension, they must be isomorphic by Proposition 3.70 from the book.

Problem 2. (3D #6) (10 points) Suppose that W is finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that $\ker(S) = \ker(T)$ if and only if there exists an invertible $P \in \mathcal{L}(W)$ such that $S = PT$.

Solution. Suppose $\ker(S) = \ker(T)$. Then the Rank-Nullity Theorem tells us that since $\text{rank}(S) + \dim \ker(S) = \text{rank}(T) + \dim \ker(T) = \dim(V)$, we have $\text{rank}(S) = \text{rank}(T) = k$. let $Sv_1, Sv_2, ..., Sv_k$ be a basis for $\text{img}\, S$, and similarly let $Tv_1, Tv_2, ..., Tv_k$ be a basis for $\text{img}\, T$.If $\dim W = m$, we can extend both bases to have $Sv_1, Sv_2, ...Sv_k, s_{k+1}, ..., s_m$ be a basis of W, and $Tv_1, T_{v-2}, ..., Tv_k, t_{k+1}, ..., t_m$ be a basis of W. Now define $P : W \to W$ as $P(Tv_i) = Sv_i, P(t_j) = s_j$. It can be shown that P is linear and surjective and $PT = S$. Since P is an operator and surjective, it is invertible.

Now suppose there exists an invertible $P \in \mathcal{L}(W)$ such that $S = PT$. Then for $v \in \ker(T), Sv = P(Tv) = P(0) = 0$ because P is linear. Hence $v \in \ker(S) \implies \ker(T) \subseteq \ker(S)$. Also, for $u \in \ker(S), Tv = P^{-1}(S(v)) = P^{-1}(v) = 0$, so $u \in \ker(T)$ and $\ker(S) \subseteq \ker(T)$. Thus $\ker(S) = \ker(T)$.

Problem 3. (5A #20) (8 points) Define the left shift operator $L \in \mathcal{L}(\mathbb{F}^\infty)$ by

$$L(z_1, z_2, z_3, \ldots) = (z_2, z_3, \ldots) .$$

(a) Show that every element of $\mathbb{F}$ is an eigenvalue of L.

Solution. For any $\lambda \in \mathbb{F}$, consider the vector $v = (1, \lambda, \lambda^2, ...)$. Then

$$L(v) = (\lambda, \lambda^2, \lambda^3, ...) = \lambda v$$

thus λ is an eigenvalue of L.

(b) Find all eigenvectors of L.

Solution. From (a), we know that each $\lambda \in \mathbb{F}$ is an eigenvalue of L. Thus for each eigenvector $v = (v_1, v_2, v_3, ...) \neq 0$ corresponding to some λ, we have

$$Tv = T(v_1, v_2, v_3, ...) = (v_2, v_3, v_4, ...) = \lambda v = (\lambda v_1, \lambda v_2, \lambda v_3, ...).$$

So we have that $v_2 = \lambda v_1, v_3 = \lambda v_2$, and more generally, $v_{k+1} = \lambda(v_k)$ for $k \geq 1$. This holds iff $v_1 = r \in \mathbb{F} \neq 0$ and $v_{k+1} = \lambda^k r$ for $k \geq 1$. Thus all eigenvectors of L are of the form $(r, r\lambda, r\lambda^2, ...)$ where r is any nonzero element of $\mathbb{F}$ and $\lambda \in \mathbb{F}$.

Problem 4. (5A #21) (8 points) Suppose $T \in \mathcal{L}(V)$ is invertible.

(a) Suppose $\lambda \in \mathbb{F}$ with $\lambda \neq 0$. Prove that λ is an eigenvalue of T if and only if $1/\lambda$ is an eigenvalue of T^{-1}.

Solution. Suppose $\lambda \in \mathbb{F}$ with $\lambda \neq 0$ is an eigenvalue of T. Then there exists a nonzero $v \in V$ such that $T(v) = \lambda v$. Applying T^{-1} to both sides, we get that

$$T^{-1}T(v) = (T^{-1}T)(v) = Iv = v = 1/\lambda(\lambda v) = T^{-1}(\lambda v)$$

thus λv is an eigenvector of T^{-1} with eigenvalue $1/\lambda$. Note $\lambda v \neq 0$ so $1/\lambda$ is an eigenvalue of T^{-1}.

Conversely, Suppose $1/\lambda \in \mathbb{F}$ with $\lambda \neq 0$ is an eigenvalue of T^{-1}. Then there exists a nonzero $v \in V$ such that $T^{-1}(v) = (1/\lambda)v$. Applying T to both sides, we get that

$$TT^{-1}(v) = (TT^{-1]})(v) = Iv = v = \lambda(1/\lambda)v = T((1/\lambda)v)$$

thus $(1/\lambda)v$ is an eigenvector of T with eigenvalue λ. Thus λ is an eigenvalue of T.

(b) Prove that T and T^{-1} have the same eigenvectors.

Solution. Suppose T has eigenvector v with eigenvalue λ, so $Tv = \lambda v$. Note that $\lambda \neq 0$ because T is an invertible operator and thus injective. Then we showed in part (a) that this means that λv is an eigenvector of T^{-1} with eigenvalue $1/\lambda$. But since the eigenspace $E(T^{-1}, 1/\lambda)$ is a vector space, v is also an eigenvector of T^{-1} with eigenvalue $1/\lambda$ because it is a scalar multiple of λv. Thus the eigenvectors of T are eigenvectors of T^{-1}.

Conversely, suppose T^{-1} has eigenvector v with eigenvalue λ, so $T^{-1}v = \lambda v$. Note that $\lambda \neq 0$ because T^{-1} is an invertible operator and thus injective. Since $(T^{-1})^{-1} = T$, we showed in part (a) that this means that λv is an eigenvector of T with eigenvalue $1/\lambda$. But since the eigenspace $E(T, 1/\lambda)$ is a vector space, v is also an eigenvector of T with eigenvalue $1/\lambda$. Thus the eigenvectors of T^{-1} are eigenvectors of T, and so T and T^{-1} have the same eigenvectors.

Problem 5. (5B #11) (10 points) Suppose V is a 2-dimensional vector space, $T \in \mathcal{L}(V)$, and the matrix of T with respect to some basis of V is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(a) Show that $T^2 - (a+d)T + (ad - bc)I = 0$.

Solution. We write

$$\begin{aligned} T^2 - (a+d)T + (ad-bc)I &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} - (a+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix} + (ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} a^2+bc & ab+bd \\ ac+dc & bc+d^2 \end{pmatrix} - \begin{pmatrix} a^2+ad & ab+bd \\ ac+dc & ad+d^2 \end{pmatrix} + \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} \end{aligned}$$

which is indeed 0 when we add and subtract the corresponding entries together.

(b) Show that the minimal polynomial of T is

$$\begin{cases} z-a & \text{if } b=c=0 \text{ and } a=d, \\ z^2-(a+d)z+(ad-bc)=0 & \text{otherwise.} \end{cases}$$

Solution. Suppose $b=c=0$ and $a=d$. Then the matrix for T is $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ and thus $p(z)=z-a$, which has degree 1, satisfies $p(T)=T-aI=0$. Since 1 is the minimal degree, $z-a$ is the minimal polynomial of T.

Now suppose it does not hold that $b=c=0$ and $a=d$. We showed in part (a) $p(z)=z^2-(a+d)z+(ad-bc)$ satisfies $p(T)=T^2-(a+d)T+(ad-bc)I=0$, thus the degree of the minimal polynomial of T is at most 2. If this degree was less than 2, the minimal polynomial of T would be of the form $p(z)=z-\lambda$ for some λ and

$$p(T)=T-\lambda I=\begin{pmatrix} a & b \\ c & d \end{pmatrix}-\lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}=0$$

which implies $b=c=0, a=d=\lambda$ which is a contradiction of our assumptions. Thus the minimal polynomial of T has to have degree 2 and so $p(z)=z^2-(a+d)z+(ad-bc)$ is the minimal polynomial of T in this case.

Problem 6. (5A #30) (5 points) Suppose $T\in\mathcal{L}(V)$ and

$$(T-2I)(T-3I)(T-4I)=0\,.$$

Suppose λ is an eigenvalue of T. Prove that $\lambda=2$ or $\lambda=3$ or $\lambda=4$.

Solution. If λ is an eigenvector of T, then for some eigenvector v corresponding to λ,

$$(T-2I)v=\lambda v-2v=(\lambda-2)v,$$
$$(T-3I)v=\lambda v-3v=(\lambda-3)v$$
$$(T-4I)v=\lambda v-4v=(\lambda-4)v.$$

Thus if $(T-2I)(T-3I)(T-4I)v=(\lambda-2)(\lambda-3)(\lambda-4)v=0$ we must have that $\lambda-2=0$ or $\lambda-3=0$ or $\lambda-4=0$, since v is nonzero. Thus $\lambda=2$ or $\lambda=3$ or $\lambda=4$.