1. (5 points) Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that the minimal polynomial of T has degree at most $1 + \dim(\operatorname{im}(T))$.

Proof.

Let $p(x)$ be the minimal polynomial for T restricted to $\operatorname{im}(T) \subset V$. Then it follows that

$$\deg(p) \leq \dim(\operatorname{im} V).$$

We claim that the minimal polynomial of T divides the polynomial $q(x) = xp(x)$. Then it will follow that for the minimal polynomial $m(x)$ of T,

$$\deg(m) \leq \deg(q) = 1 + \deg(p) \leq 1 + \dim(\operatorname{im} V)),$$

so we are done. By the rank-nullity theorem, we can factor any $v \in V$ as

$$v = u + w,$$

where $u \in \ker V$, and $w \in \operatorname{im} V$. Then we know that

$$q(T)(v) = q(T)(u) + q(T)(v) = 0 + 0 = 0,$$

because $u \in \ker T \implies Tu = 0$, and $w \in \operatorname{im} T \implies p(T)w = 0$, and so $q(T)u = q(T)w = 0$. Therefore for all $v \in V$, $q(T)v = 0$, and thus the minimal polynomial of T must divide $q(x)$. □

Remark: You cannot assume that the base field is algebraically closed. In particular, the minimal polynomial of T may not factor into linear terms in this case.

2. (6 points) Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T is invertible iff $I \in \text{span}(T, T^2, \cdots, T^{\dim(V)})$.

Proof.
Let $n = \dim V$, and write the minimal polynomial for T be $m(x)$. By the Cayley-Hamilton theorem, we can write the characteristic polynomial of T as

$$p(x) = a_n x^n + \cdots + \det(T),$$

and in particular

$$a_n T^n + \cdots + \det(T) I = 0.$$

First, suppose T is invertible. Then since $\det(T) \neq 0$, we can write

$$I = -\frac{1}{\det(T)}(a_n T^n + \cdots + a_1 T),$$

and so it follows that $I \in \text{span}(T, T^2, \cdots, T^{\dim(V)})$.

To prove the other direction, suppose $I \in \text{span}(T, T^2, \cdots, T^{\dim(V)})$, and write

$$I = c_1 T + c_2 T^2 + \cdots + c_n T^n.$$

Then for any $v \in \ker T$, we know that

$$v = Iv = c_1 Tv + c_2 T^2 v + \cdots + c_n T^n v = 0,$$

and so it follows that $\ker T = 0$, and so T is invertible.

□

3. (5 points) Suppose $\mathbb{F} = \mathbb{C}$, V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that V has a k-dimensional T-invariant subspace for each $k = 1, \cdots, \dim(V)$.

Proof.
We know that there is a basis $\{v_1, \cdots, v_n\}$ of V for which T has an upper-triangular matrix. Then since the matrix is upper triangular, the subspace generated by the eigenvectors $\{v_k, \cdots, v_n\}$ is T-invariant. □

4. (6 points) Suppose $\mathbb{F} = \mathbb{C}$, V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that there exists a basis of V with respect to which T has a lower-triangular matrix.

   Proof. We know that there is a basis $\{v_1, \cdots, v_n\}$ of V for which T has an upper-triangular matrix. Check that for the basis $\{v_n, \cdots, v_1\}$, T has a lower-triangular matrix. $\square$

5. (7 points) Suppose $\mathbb{F} = \mathbb{C}$, V is finite-dimensional and $T \in \mathcal{L}(V)$.

   a. Prove that if $T^4 = I$, then T is diagonizable.
   b. Prove that if $T^4 = T$, then T is diagonizable.
   c. Give an example of $T \in \mathcal{L}(\mathbb{C}^2)$ such that $T^4 = T^2$ and T is not diagonizable.

Proof.

(a) To show that T is diagonizable, it is enough to show that the minimal polynomial of T factors into linear terms with multiplicity 1. Since $T^4 = I$, it follows that the minimal polynomial of T has to divide

$$z^4 - 1 = (z-1)(z+1)(z-i)(z+i).$$

And so it follows that the minimal polynomial of T has to factor into linear terms with multiplicity one.

(b) Same argument as (a), check that

$$z^4 - z = z(z-1)(z - \exp(\frac{2\pi i}{3}))(z - \exp(\frac{4\pi i}{3})).$$

(c) Let $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Check that $T^4 = T^2 = I$. Since T only has 0 as a repeated eigenvalue, but its eigenspace is only spanned by the vector $(0,1)$, T is not diagonizable. □

6. (12 points)
The Fibonacci sequence $F_0, F_1, \cdots$ is defined by $F_0 = 0, F_1 = 1$, and $F_n = F_{n-2} + F_{n-1}$ for $n \geq 2$. Define $T \in \mathcal{L}(\mathbb{R}^2)$ by $T(x, y) = (y, x + y)$.

   a. Show that $T^n(0, 1) = (F_n, F_{n+1})$ for each $n \in \mathbb{Z}_{\geq 0}$.
   b. Find the eigenvalues of T.
   c. Find a basis of $\mathbb{R}^2$ consisting of eigenvectors of T.
   d. Show that
$$F_n = \frac{1}{\sqrt{5}}[(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n].$$

Proof.

(a) Proof by induction, check that

$$T(F_{n-1}, F_n) = (F_n, F_{n-1} + F_n) = (F_n, F_{n+1}).$$

(b) The characteristic polynomial of T is $x^2 - x - 1$, so the eigenvalues are

$$\lambda^2 - \lambda - 1 = 0 \iff \lambda = \frac{1 \pm \sqrt{5}}{2}.$$

(c) Check that the eigenvalues for $\lambda_{\pm} = \frac{1\pm\sqrt{5}}{2}$ are given by

$$v_{\pm} = (1, \frac{1 \pm \sqrt{5}}{2}).$$

(d) For the change of basis matrix B, we can write

$$T = B\Lambda B^{-1},$$

for $\Lambda = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix}$. Thus, we know that

$$(F_n, F_{n+1}) = T^n(0, 1) = (B\Lambda B^{-1})^n(0, 1) = B\Lambda^n B^{-1}(0, 1).$$

Since we know that

$$\Lambda^n = \begin{pmatrix} \frac{1+\sqrt{5}}{2}^n & 0 \\ 0 & \frac{1-\sqrt{5}}{2}^n \end{pmatrix},$$

the value of F_n can now be determined by a simple calculation in matrix multiplication. □