1. (7 points) Suppose $T \in L(V)$ and U is a subspace of V. Prove that

$$U \text{ is } T\text{−invariant} \iff U^{\perp} \text{ is } T^{*}\text{−invariant.}$$

Proof.
For any $u \in U$, $w \in U^{\perp}$, we have that

$$\langle Tu, w\rangle = \langle u, T^{*}w\rangle.$$

By definition, U is T-invariant if for any $u \in U$, the image Tu is also in U. We know that $Tu \in U$ is equivalent to $\forall w \in U^{\perp}$, $\langle Tu, w\rangle = 0$. Likewise, $Tw \in U^{\perp}$ is equivalent to $\forall u \in U$, $\langle u, T^{*}w\rangle = 0$. Now, since

$$\langle Tu, w\rangle = 0 \iff \langle u, T^{*}w\rangle = 0,$$

we have the chain of equivalent statements

$$\begin{aligned} U \text{ is } T\text{−invariant} &\iff \forall u \in U, Tu \in U \\ &\iff \forall w \in U^{\perp}, \forall u \in U, \langle Tu, w\rangle = 0 \\ &\iff \forall w \in U^{\perp}, \forall u \in U, \langle u, T^{*}w\rangle = 0 \\ &\iff \forall w \in U^{\perp}, T^{*}w \in U^{\perp} \\ &\iff U^{\perp} \text{ is } T^{*}\text{−invariant.} \end{aligned}$$

□

2. (6 points) Suppose $T \in L(V, W)$. Suppose $e_1, ..., e_n$ is an orthonormal basis of V and $f_1, ..., f_m$ is an orthonormal basis of W. Prove that
$$\|T(e_1)\|^2 + \cdots + \|T(e_n)\|^2 = \|T^*(f_1)\|^2 + \cdots + \|T^*(f_m)\|^2 .$$

Proof.
Since $\{f_j\}$ is an orthonormal basis of W, we have
$$\|T(e_i)\|^2 = \sum_j \langle T(e_i), f_j \rangle^2,$$
and also since $\{e_i\}$ is an orthonormal basis of V, we have
$$\|T^*(f_j)\|^2 = \sum_i \langle e_i, T^*(f_j) \rangle^2.$$
We sum the first equation over i, and the second equation over j and get
$$\begin{aligned}\|T(e_1)\|^2 + \cdots + \|T(e_n)\|^2 &= \sum_i \sum_j \langle T(e_i), f_j \rangle^2 \\ &= \sum_j \sum_i \langle e_i, T^*(f_j) \rangle^2 \\ &= \|T^*(f_1)\|^2 + \cdots + \|T^*(f_m)\|^2 .\end{aligned}$$
□

3. (7 points) Let $F = \mathbb{C}$ and suppose $T \in L(V)$ is normal. Show that T is self-adjoint if and only if all the eigenvalues of T are real.

   Proof.
   We learned in class that every self-adjoint operator has only real eigenvalues. So we only need to show the converse.

   By the spectral theorem over $\mathbb{C}$, we can find an orthonormal basis of V such that the matrix A representing the operator T with respect to this basis is diagonal. Since the diagonal entries are the eigenvalues which are assume to be all real numbers, it follows that $A = A^*$, and so we conclude that $T = T^*$.

   □

4. (6 points) Let $F = \mathbb{C}$ and suppose $T \in L(V)$ is a normal operator such that $T^9 = T^8$. Prove that T is self-adjoint and $T^2 = T$.

   Proof. Since $T^9 - T^8 = 0$, we know that the minimal polynomial of T divides $x^9 - x^8$. In particular, all the eigenvalues λ of T satisfy $\lambda^9 = \lambda^8$. In particular, we know that $\lambda = 0, 1$ and thus all eigenvalues of T are real. By the previous exercise, we may conclude that T is self-adjoint.

   Because of the spectral theorem for $\mathbb{C}$, we know that T is diagonizable. Since the possible eigenvalues of T are $0, 1$, its minimal polynomial must divide $x(x-1)$, and thus $T^2 = T$. □

5. (6 points) Let n be a positive integer and $T \in \mathcal{L}(\mathbb{F})$ be the operator whose matrix with respect to the standard basis consists of all 1s. Show that T is a positive operator.

Proof. A positive operator is a self-adjoint operator whose eigenvalues are all non-negative. Since the matrix representing T is self-adjoint, it follows that T is also self-adjoint. Now since the matrix representing T has eigenvalues $n, 0$, it follows that T is a positive operator. □

6. (4 points) Give an example of $T \in L(\mathbb{C}^2)$ such that 0 is the only eigenvalue of T and the singular values of T are $5, 0$.

Proof. Think of the matrix

$$A = \begin{pmatrix} 0 & 0 \\ 5 & 0 \end{pmatrix}.$$

We have that

$$A^*A = \begin{pmatrix} 0 & 5 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 5 & 0 \end{pmatrix} = \begin{pmatrix} 25 & 0 \\ 0 & 0 \end{pmatrix},$$

and thus the eigenvalues of A^*A are $25, 0$. The singular values of A are the square roots of the eigenvalues of A^*A, which are $5, 0$. On the other hand, one sees that the equation determining the eigenvalues of A

$$\begin{pmatrix} 0 & 5 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix},$$

is equivalent to $5y = \lambda x$, $0 = \lambda y$, which has a nonzero solution (x, y) if and only if $\lambda = 0$, in which case $(x, y) = (1, 0)$ is a nontrivial solution.

Thus we conclude that 0 is the only eigenvalue of A. □